

A trivial observation on time reversal in random matrix theory

L. Kaplan¹, F. Leyvraz², C. Pineda^{2,3}, and T. H. Seligman^{2,3}

¹Department of Physics, Tulane University, New Orleans, USA

²Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, México

³Centro Internacional de Ciencias, Cuernavaca, México

E-mail: carlospgmat03@gmail.com, seligman44@yahoo.com.mx

Abstract. It is commonly thought that a state-dependent quantity, after being averaged over a classical ensemble of random Hamiltonians, will always become independent of the state. We point out that this is in general incorrect: if the ensemble of Hamiltonians is time reversal invariant, and the quantity involves the state in higher than bilinear order, then we show that the quantity is only a constant over the orbits of the invariance group on the Hilbert space. Examples include fidelity and decoherence in appropriate models.

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Whereas eigenfunctions of time reversal invariant (TRI) systems can always be chosen real, complex linear combinations of such functions display different statistical properties than real ones. While this observation is trivial, its consequences for applications of random matrix theory (RMT) have been largely ignored. We shall show that these consequences are important in properties of higher (mainly fourth) order in the wave functions, such as transition probabilities, (inverse) participation ratios, fidelity, purity, or von Neumann entropy. While these effects are often subleading in the dimension N of Hilbert space, for several quantities of physical interest, including inverse participation ratios and purity decay rates, the effects do appear at leading order. Moreover, when considering entanglement or decoherence in the context of quantum information we often deal with small Hilbert spaces, possibly even a single qubit, where the choice of a real or complex initial state becomes extremely important. Among the classical ensembles of Hamiltonians, namely the Gaussian orthogonal, unitary, and symplectic ensembles (GOE, GUE, GSE) as described by Cartan [1], GOE and GSE represent TRI systems and GUE represents non-TRI ones. In the GUE case, averaging over Hilbert space is implicit in the ensemble average, but this is not the case for the two TRI ensembles with important consequences, some of which will be discussed in this paper. The same holds for the corresponding circular ensembles of unitary matrices [2]. Among more general ensembles, such as the chiral ones, similar distinctions have to be made [3]. Note that the problems addressed involve statistics of the wave functions only, and do not concern spectral properties. Thus, the effects apply equally to TRI Gaussian and circular ensembles. To avoid needless repetitions we limit ourselves to the GOE and COE for the TRI case.

A very simple problem illustrates the type of effect we deal with. Let us look at the autocorrelation function

$$A(t) = |\langle \psi | e^{-iHt} | \psi \rangle|^2 \quad (1)$$

for a TRI Hamiltonian H drawn from a GOE. Its long time average is equal to the inverse participation ratio $\mathcal{I}_{|\psi\rangle} = \sum_{|\alpha\rangle} |\langle \alpha | \psi \rangle|^4$, where $|\alpha\rangle$ are the eigenstates of H . Since $\mathcal{I}_{|\psi\rangle}$ is a time average, we may hope that it is equal to an ensemble average, independent of the initial state $|\psi\rangle$, for a chaotic or mixing dynamics. Yet one easily finds that $N\mathcal{I} = 2N/(N+1) \xrightarrow{N \rightarrow \infty} 2$ when averaged over all (complex) states, while $N\mathcal{I} = 3N/(N+2) \xrightarrow{N \rightarrow \infty} 3$ when averaged over all real states. The result is obvious, since in the former case $\langle \alpha | \psi \rangle$ behave as random complex variables, while in the latter case they behave as random real variables. Such effects on the inverse participation ratio have been studied before in the context of wave function statistics in billiards with TRI [4]. There a comparison was made between the choice of real or complex initial state in TRI systems and the choice of a symmetric or non-symmetric initial state in a system possessing a unitary symmetry, such as parity. Furthermore, initial states were discussed that are linear combinations of real and complex random wave functions, exhibiting a transition between the two limits.

In fact, taking a long-time average is unnecessary, and the effect is already visible in the short-time dynamics of a TRI system. The average autocorrelation function, for large N , is given by

$$\langle A(t > 0) \rangle = \begin{cases} \frac{2 - b_2(t/\tau_H)}{N} & \text{for complex } |\psi\rangle, \\ \frac{3 - b_2(t/\tau_H)}{N} & \text{for real } |\psi\rangle \end{cases}. \quad (2)$$

$\langle \dots \rangle$ indicates an ensemble average, τ_H is the Heisenberg time of the Hamiltonian, and $b_2(t)$ is the two level form factor of the GOE [3]. For $0 < t \ll \tau_H$, we have simply $\langle A(t) \rangle \approx 1/N$ and $\langle A(t) \rangle \approx 2/N$ respectively, i.e. the short-time return probability for a real initial state is double that of a complex initial state in a TRI system. This weak localization effect is easily understood in semiclassical terms, since the factor of 2 results from constructive interference between each returning path and its time-reversed counterpart.

From these trivial examples we see immediately that whenever we average a quantity that is not bilinear in the wave function, the average over a TRI ensemble such as GOE or GSE (and thus the time average if ergodicity holds) does depend on whether the initial state is real (up to an overall phase) or complex.

Proposing an experiment is not altogether trivial. One possibility is to excite a solid metal block elastically with M pings at different times and places, corresponding to a state $|\psi\rangle = \sum_{j=1}^M e^{i\phi_j} |\psi_j\rangle$, where each $|\psi_j\rangle$ is real but the relative phases are random. Assuming all pings have the same strength, the average autocorrelation function at long times yields $(2 + 1/M)/N$, e.g. $2.5/N$ for two pings versus $3/N$ for a single ping. Similarly, at short times we have $(1 + 1/M)/N$. Such an experiment can be performed [5], though it might not be all too interesting as the outcome is clear.

On a slightly more formal note we may say the following: Starting from an arbitrary state $|\psi\rangle$ in some Hilbert space \mathcal{H} of dimension N , we shall cover, up to normalization, the entire Hilbert space by the orbit of $|\psi\rangle$ traced by $\mathcal{U}(N)$ on \mathcal{H} .

Consider now that the GUE is can be defined as a set of diagonal matrices Δ with the appropriate measure $d\nu(\Delta)$ composed with the unitary matrices u as $u^\dagger \Delta u$, with

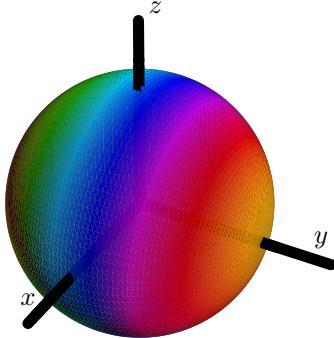


Figure 1. The Bloch sphere [6], representing the orbits of states under the action of $\mathcal{O}(2)$.

the invariant Haar measure $d\mu(u)$. It is then immediately clear that averaging over the GUE will include averaging over all states. In other words, a state dependent quantity averaged over the Hamiltonians is automatically constant throughout the Hilbert space. If, on the other hand, we consider a GOE, the corresponding representation is $o^\dagger \Delta o$ with the Haar measure of the orthogonal group $d\mu(o)$. The implicit averaging over states will then be limited to the orbits of $\mathcal{O}(N)$ on the Hilbert space. A state dependent quantity averaged over the Hamiltonians, in the TRI case, is therefore only constant on the orbits of the original state.

If we consider the circular ensembles, the situation is slightly more involved, as the symmetry operations defining these ensembles are not similarity transformations. Recall that the CUE is the unitary group $\mathcal{U}(N)$ itself and thus is left *and* right invariant under $\mathcal{U}(N)$; obviously this includes similarity transformations and thus again the ensemble average includes state averaging. For the COE the situation is more complicated as the measure is invariant under $\mathcal{U}(N)$, but if S is an element of the COE the operation is defined as $d\mu(S) = d\mu(u^t S u)$. Note that this is not a similarity transformation. Yet if we restrict $\mathcal{U}(N)$ to $\mathcal{O}(N)$ we have a similarity transformation as the transpose of an orthogonal matrix is its inverse. Thus the same orbits discussed above describe the averaging we achieve with the ensemble average of the Hamiltonians. The two-dimensional Hilbert space associated with a qubit provides the best example. Representing this space in terms of the Bloch sphere, the orbits of $\mathcal{O}(2)$ are rings around the y axis of the sphere as illustrated in figure 1.

Considering the recent interest in developing RMT models for fidelity decay [7, 8, 9] and decoherence [10, 11, 12, 13, 14], we may ask how the effect appears in this context. Clearly, the effect will not be observed for the fidelity amplitude or for coherences (off-diagonal elements of the density matrix), since these quantities are bilinear in the wave functions. On the other hand, considerations regarding the choice of initial state will be highly relevant in the study of fidelity, purity, or von Neumann entropy. Indeed initially puzzling results on purity decay for one or two qubits [14] lead to, and will be at the centre, of the present analysis. The importance of entanglement and decoherence of one- and two-qubit systems as the basic building blocks of quantum information [6] justifies this focus, particularly as manipulations of qubits using tools of quantum optics allow complex states to be produced in a very simple way. Yet we shall start with the simpler case of fidelity decay, a benchmark in quantum information.

The fidelity amplitude is defined as

$$f_\epsilon(t) = \langle \psi | e^{-iH_0 t} e^{iH_\epsilon t} | \psi \rangle \quad (3)$$

and fidelity is given as $F_\epsilon(t) = |f_\epsilon(t)|^2$, where $H_\epsilon = H_0 + \epsilon V$. Here $|\psi\rangle$ is any function, H_0 is any Hamiltonian of interest, V a perturbation, and ϵ a real parameter. In [8] both H_0 and V were modeled by random matrix ensembles. While in some special cases exact results were obtained [15, 16, 17], linear response results were obtained in a wide variety of cases (see [12] and references therein). In [8, 12] the linear response result for $F_\epsilon(t)$ is given as

$$\langle F(t) \rangle = \langle |f_\epsilon(t)|^2 \rangle = \langle f_\epsilon(t) \rangle^2 + (2\pi\epsilon)^2 (2/\beta_V) \mathcal{I} t^2 + O(\epsilon^4), \quad (4)$$

where \mathcal{I} is the inverse participation ratio of $|\psi\rangle$ in the eigenbasis of H_0 and $\beta_V = 1, 2, 4$ specifies whether the perturbation is taken from a GOE, GUE, or GSE. The validity of this approximation can be extended by exponentiating the second order term. The conclusion in these papers was that for a random state $\mathcal{I} = O(1/N)$, and thus $F_\epsilon \approx \langle f_\epsilon \rangle^2$. If we further average over random states, this is still correct. Yet we can reasonably ask what the state-averaged fidelity would be if the average is limited to some small subspace of the total Hilbert space, which for some reason is interesting or experimentally accessible. Then the inverse participation ratio is large and we observe an effect of order one exactly like the one discussed above. The result without state average will depend on the initial state if we allow complex states and consequently self averaging is lost. In this example the choice of the subspace over which we average might be somewhat arbitrary, though often only certain frequency regions are accessible.

For a composite system, the situation is different. We often have a natural separation between a smaller system, which we call the central system, and an environment that interacts with it. Typically we will only be interested in the central system, or only the central system may be accessible to experiment. Such is the case for one or a few qubits coupled to an environment, where both the environment and the coupling are modeled by random matrix ensembles [13, 14].

Purity of a density matrix ρ is defined as $P(\rho) = \text{tr } \rho^2$, and is a measure of the degree of mixedness of the density matrix, or the degree of entanglement of a central system with an environment. Thus it is also a measure of decoherence. Consider a single qubit and an environment evolving under the Hamiltonian

$$H = H_e + \lambda V_{e,q} \quad (5)$$

where H_e acts on an environment of dimension N_e , $V_{e,q}$ is a $2N_e \times 2N_e$ matrix coupling the qubit and the environment, and λ a parameter controlling the strength of the coupling. The initial state is a pure separable state. Let us choose both H_e and $V_{e,q}$ from the GOE. The resulting ensemble of Hamiltonians is invariant under local orthogonal transformations. Evaluating purity of the qubit density matrix, in linear response approximation, we obtain for large N_e [14]:

$$P(t) = 1 - \lambda^2 \left\{ t^2 [3 - \cos(2\gamma)] + 2t\tau_H - 2B_2(t) \right\}, \quad (6)$$

where

$$B_2(t) = 2 \int_0^t d\tau \int_0^\tau d\tau' b_2(\tau'/\tau_H) \quad (7)$$

is the double integral of the two-level GOE form factor and τ_H is the Heisenberg time of the environment. γ is the angle between the xz plane and the representation in the

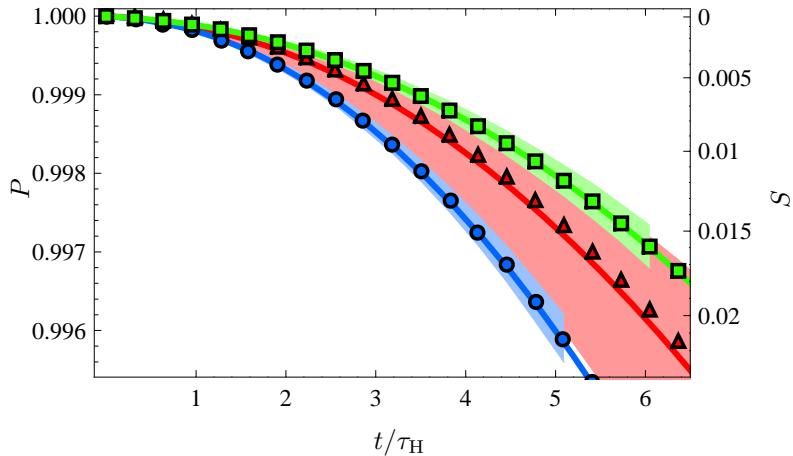


Figure 2. We display the behaviour of purity and von Neumann entropy, for initial qubit states in different regions in the Bloch sphere, fixing $\lambda = 10^{-3}$ and environment size $N_e = 1024$. The coding is as follows: green (squares) for $\gamma = 0$, blue (circles) for $\gamma = \pi/2$, and red (triangles) for arbitrary γ . For each case, the calculation was repeated 100 times with randomly chosen realizations of H_e and $V_{e,q}$, and random initial states. The shaded coloured regions represent envelopes encompassing all 100 runs, the average purity is indicated by symbols, and the predicted behaviour of (6) is shown by solid curves.

Bloch sphere of the initial state of the qubit. As, for a single qubit, purity and von Neumann entropy $S = -\text{tr} \rho \log \rho$ have a one to one relation, (6) can be translated easily to obtain entropy increase.

In figure 2 we show $P(t)$ for $\gamma = 0$ (green squares), for $\gamma = \pi/2$ (blue circles), and for random initial states in the whole Bloch sphere (red triangles). In contrast to the GUE case, the average purity depends on the initial state (via the angle γ). The fastest decay of purity is observed for $\gamma = \pi/2$, where the state is orthogonal to its time reversal image. The slowest decay is observed for $\gamma = 0$, which characterizes TRI states. In figure 3 we show numerical results for the standard deviation of the purity as a function of N_e , the dimension of the Hilbert space of the environment. We consider the same cases as in figure 2. Note that H_e , $V_{e,q}$, and the initial state of the environment are randomly chosen from their respective ensembles. We see that for fixed γ , the standard deviation falls off as $N_e^{-1/2}$. By contrast, the standard deviation converges to a finite value when γ is unrestricted. Since, for $N_e \rightarrow \infty$, the variations in $\cos 2\gamma$ are the only source of purity fluctuations, the standard deviation of the purity is

$$\sigma_P = \frac{4}{3\sqrt{5}}\lambda^2 t^2 + O(\lambda^4, N_e^{-1}). \quad (8)$$

This value is plotted in figure 3.

This result for a single qubit is especially significant in view of the fact that the decoherence of several qubits can often (in the high purity approximation) be reduced to the case of a single qubit [18].

Summarizing, we have shown that, for the general (non-TRI) case, averaging over the ensemble of Hermitian Hamiltonians (GUE) implies a full average over all states.

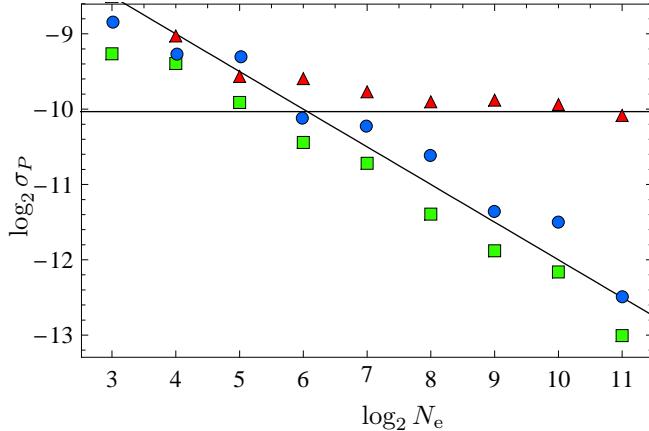


Figure 3. We plot the standard deviation of purity σ_P at time $t = 40$, as a function of the environment dimension, using the same coding and value of λ as in the previous figure. For a fixed value of γ (blue circles and green squares), there is asymptotic self averaging, as indicated by the line $\propto 1/\sqrt{N_e}$. In contrast, for arbitrary initial conditions (red triangles), the standard deviation at large N_e approaches the finite value predicted in (8), here plotted as a horizontal line.

For TRI systems, on the other hand, if a state-dependent quantity is averaged over the ensemble of real Hamiltonians (GOE), it will in general still depend on the orbit of the initial state under the orthogonal group. We have shown that this actually happens, if the averaged quantity depends on the state in higher than bilinear order. The variance of fidelity, in particular, shows such behaviour, but it is of order $1/N$, where N is the dimension of the system, and thus often insignificant. We have further displayed a specific TRI random matrix model for the decoherence of a qubit, for which the effect is of order one.

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